

# ON THE DERIVATIVES OF HARMONIC FUNCTIONS ON THE BOUNDARY\*

BY

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**Abstract.** Let  $U$  be harmonic in a closed region  $R$ , whose boundary contains a regular surface element  $E$ , with  $\alpha$  representation  $z = \phi(x, y)$ . If  $E$  has bounded curvatures, and if  $\phi(x, y)$  and the boundary values of  $U$  on  $E$  have continuous derivatives of order  $n$  which satisfy a Dini condition, then the partial derivatives of  $U$  of order  $n$  exist, as limits, on  $E$ , and are continuous in  $R$  at any interior point of  $E$ . Hölder conditions on the boundary values of  $U$ , or on their derivatives of order  $n$ , imply Hölder conditions on  $U$ , or the corresponding derivatives, in  $R$ , in the neighborhood of the interior points of  $E$ .

1. **Introduction.** A large number of articles contain studies of the existence and behavior of the limits of the derivatives, on the boundary, of harmonic functions, when these are given as the potentials of various spreads of attracting matter. On the other hand, studies of the derivatives of harmonic functions defined directly by their boundary values are surprisingly few, particularly in space of three dimensions. In a paper of my own,<sup>†</sup> the problem for the logarithmic potential has been investigated. In space, there are few actual results on derivatives of order higher than the first, and the conditions imposed on the boundary values are much heavier than need be.

The method used in previous work has been to express the given harmonic function as the potential of a double distribution, through a Neumann series. While this method has not yet yielded the results of which it is capable, it contains an element of indirectness, in that the conditions on the boundary values must first be translated into conditions on the moment of the double distribution, and from these, the behavior of the derivatives of the harmonic function must then be inferred. The method here used is based on Poisson's integral, applied to a sphere internally tangent to the boundary. In the case of the derivatives of the first order, this method requires more than is necessary for the theorems, for in order to apply it, we must assume that spheres, internally tangent to the boundary, and containing no exterior

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† *Harmonic functions and Green's integral*, these Transactions, vol. 13 (1912), pp. 109–132. References to the literature are given there, and in my two previous papers, *ibid.*, vol. 9 (1908), pp. 39–66.

points, exist. For derivatives of higher order, however, this requirement ceases to be extraneous. As the Neumann method is comparatively simple for derivatives of the first order, the two procedures appear to complement each other nicely.

The results, in their generality, for derivatives of higher order are new. Those for derivatives of the first order are in every respect more general than any at hand, with the exception of Liapounoff's,\* who requires less of the boundary surface, but more of the boundary values. The results here obtained with respect to Hölder conditions appear to be new for  $n > 1$ , and those for  $U$  itself are more general than those at hand.† As an incidental result, a simple proof is given of the analytic character of harmonic functions.‡

2. The derivatives of first order of Poisson's integral. Let  $U$  be harmonic in a sphere of radius  $a$ . We consider first its derivative in the direction of its polar axis,  $\theta = 0$ , at a point of that axis. Writing Poisson's integral in the form

$$U(\rho) = \frac{a(a^2 - \rho^2)}{2} \int_0^\pi \frac{f(\theta) \sin \theta}{r^3} d\theta, \quad f(\theta) = \frac{1}{2\pi} \int_0^{2\pi} U(a, \phi, \theta) d\phi,$$

$$r^2 = a^2 + \rho^2 - 2a\rho \cos \theta,$$

we find, for  $\rho < a$ ,

$$\frac{\partial U}{\partial \rho} = -a\rho J_1 - \frac{3a(a + \rho)}{2} J_2,$$

where

$$J_1 = \int_0^\pi \frac{f(\theta) \sin \theta d\theta}{r^3}, \quad J_2 = (a - \rho) \int_0^\pi \frac{f(\theta)(\rho - a \cos \theta) \sin \theta d\theta}{r^5}.$$

Our task is to show that these integrals approach limits as  $\rho \rightarrow a$ , under suitable conditions on  $f(\theta)$ , and to observe something as to the rate of approach. Assuming the existence of  $f'(\theta)$  near  $\theta = 0$ , it can be shown, by an integration

\* *Sur certaines questions qui se rattachent au problème de Dirichlet*, Journal de Mathématiques, (5), vol. 4 (1898), p. 241. To the literature cited in my papers referred to above, should be added Korn, *Mathematische Annalen*, vol. 53 (1900), pp. 593–608; P. Lévy, *Sur l'allure des fonctions de Green et de Neumann dans le voisinage du contour*, *Acta Mathematica*, vol. 42 (1920), pp. 207–267.

† See Korn, *Sur les équations de l'élasticité*, *Annales de l'Ecole Normale*, (3), vol. 24 (1907), pp. 23, 25.

Since the writing of this paper, I have learned of one by Schauder, *Potentialtheoretische Untersuchungen, Erste Abhandlung*, about to appear in the *Mathematische Zeitschrift*. The contents of the two papers are confined to results on Hölder conditions on  $U$  and its derivatives of the first order. Those for  $U$  itself are essentially the same; for the derivatives of first order, Schauder's are more general than mine, in that bounded curvatures of the bounding surface are not required.

‡ I wish to acknowledge my indebtedness to my colleague, Dr. Gergen, for his careful examination of the manuscript.

by parts, that the normal derivative of  $U$  approaches a limit provided  $f'(\theta)$  satisfies a condition of the type used by Dini,\* namely that the integral

$$\int_0^\eta \frac{|f'(\theta)|}{\theta} d\theta$$

is convergent. For our purposes, however, a somewhat different condition will be used. We shall show, namely, that the normal derivative of  $U$  has a limit provided  $f(\theta)$  is integrable and bounded, and such that the integral

$$\int_0^\eta \frac{|f(\theta) - f(0)|}{\theta^2} d\theta$$

is convergent.

It is legitimate to assume  $f(0) = 0$ , since the subtraction of a constant from  $U$  affects neither its derivatives nor the validity of the hypotheses. With a number  $\eta$ ,  $0 < \eta \leq \pi/2$ , we break up the integrals  $J_1$  and  $J_2$  each into two,

$$J_1 = J_{11} + J_{12}, \quad J_2 = J_{21} + J_{22},$$

$J_{11}$  and  $J_{21}$  being extended over the interval  $(0, \eta)$ , and  $J_{12}$  and  $J_{22}$ , over the interval  $(\eta, \pi)$ . Then, for any fixed  $\eta$ , the functions  $J_{12}$  and  $J_{22}$  are analytic in  $\rho$  at  $\rho = a$ , and hence have limits from which they differ arbitrarily little for all  $\rho$  sufficiently near  $a$ . Hence, if it can be shown that  $\eta$  can be so restricted that  $J_{11}$  and  $J_{21}$  are arbitrarily small in absolute value, independently of  $\rho$ , the existence of a limit for the derivative of  $U$  will be established.

But this is immediate. From the equations

$$\begin{aligned} r^2 &= (a - \rho)^2 + 4a\rho \sin^2 \frac{\theta}{2} = (a - \rho \cos \theta)^2 + \rho^2 \sin^2 \theta \\ &= (a \cos \theta - \rho)^2 + a^2 \sin^2 \theta, \end{aligned}$$

we derive the inequalities

$$\frac{a - \rho}{r} \leq 1, \quad \frac{|\rho - a \cos \theta|}{r} \leq 1, \quad \frac{\theta}{r} \leq \frac{\pi \sin \theta}{2r} \leq \frac{\pi}{2a},$$

the last holding for  $0 \leq \theta \leq \eta$ , since  $\eta \leq \pi/2$ . Using them, we find

$$|J_{11}| \leq \left(\frac{\pi}{2a}\right)^3 \int_0^\eta \frac{|f(\theta)|}{\theta^2} d\theta, \quad |J_{21}| \leq \left(\frac{\pi}{2a}\right)^3 \int_0^\eta \frac{|f(\theta)|}{\theta^2} d\theta.$$

The integrals are convergent, by hypothesis, and so approach 0 with  $\eta$ . As they are independent of  $\rho$ , the existence of the limits of  $J_1$  and  $J_2$ , and so of the normal derivative of  $U$ , is established. We note, moreover, that for fixed

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\* Acta Mathematica, vol. 25 (1902), p. 224.

$\eta$ , the derivatives of  $J_{12}$  and  $J_{22}$  with respect to  $\rho$  are bounded in absolute value by a number depending only on the bound for  $|f(\theta)|$ , so that we may enunciate the results as follows:

**THEOREM I.** *Let  $U$  be harmonic in the sphere of radius  $a$ , and be given by Poisson's integral with the bounded integrable boundary values  $U(a, \phi, \theta)$ . Let the average of these values on parallel circles,*

$$f(\theta) = \frac{1}{2\pi} \int_0^{2\pi} U(a, \phi, \theta) d\phi,$$

*be subject to the requirement that the integral*

$$(1) \quad \int_0^\eta \frac{|f(\theta) - f(0)|}{\theta^2} d\theta, \quad 0 < \eta \leq \frac{\pi}{2},$$

*be convergent. Then the derivative of  $U$  in the direction of the polar axis  $\theta=0$  approaches a limit at the surface of the sphere for approach along the polar axis. Moreover, the approach to the limit is uniform for any class of boundary functions which are uniformly bounded in absolute value, and for which the integral (1) approaches 0 uniformly with  $\eta$ .*

**Tangential derivatives.** A similar theorem exists for the tangential derivatives. The derivative of  $U$  in the direction of increasing  $\theta$  in the meridian half-plane  $\phi = \phi_0$ , at a point of the polar axis, is given by

$$\frac{1}{a} \frac{\partial U}{\partial \theta} = \frac{3a\rho(a^2 - \rho^2)}{2} \int_0^\pi \frac{F(\theta) \sin^2 \theta}{r^5} d\theta,$$

$$F(\theta) = \frac{1}{2\pi} \int_0^{2\pi} U(a, \phi, \theta) \cos(\phi - \phi_0) d\phi.$$

The same reasoning as that just employed then leads to

**THEOREM II.** *Theorem I holds also for the tangential derivatives of  $U$ , provided the function  $F(\theta)$  satisfies the conditions there imposed on  $f(\theta)$ .*

**Remark.** Even if  $f(\theta)$  and  $F(\theta)$  have continuous derivatives of the first two orders, the conditions of Theorems I and II will not be fulfilled, unless these functions, and their first derivatives, vanish at  $\theta=0$ . This difficulty, however, may at once be met by the subtraction from  $U$  of a linear function, tangent to  $U$  at  $\theta=0, \rho=a$ . The theorems are therefore more general than at first appears.

**Limiting values.** Under the hypotheses imposed on  $f(\theta)$  and  $F(\theta)$ , it will be seen that the limiting values of the normal and tangential derivatives of  $U$  are given by the convergent integral

$$(2) \quad \frac{\partial U}{\partial \rho} = - \frac{1}{4a} \int_0^\pi \frac{[f(\theta) - f(0)] \cos \frac{\theta}{2}}{\sin^2 \frac{\theta}{2}} d\theta,$$

and by the derivatives in the same direction of the boundary values of  $U$ , respectively.

3. **Formulation of conditions which insure the existence and continuity of derivatives on the boundary.** An advantage of the present method of study is its local character. If we wish to consider the behavior of the derivatives of a harmonic function  $U$  only on a portion of the surface  $S$  bounding the region in which it is given, we need make special hypotheses on  $S$  and the boundary values of  $U$  only on this portion. Accordingly, we shall deal with a region whose boundary contains a *regular surface element*  $E$ , that is, a set of points, which is given, for a suitable orientation of the coordinate axes, by an equation

$$z = \phi(x, y),$$

$\phi(x, y)$  being one-valued, and having continuous derivatives of the first order, for  $(x, y)$  in a closed regular region of the  $(x, y)$ -plane. A regular region of the plane is one bounded by a regular curve, without double points.\*

We shall employ the following conditions.

CONDITION  $A_n$ .  *$R$  is a bounded open continuum, whose boundary  $S$  contains a regular surface element  $E$ , with the following properties:*

(a) *it has definite radii of curvature at each point, which are uniformly bounded;*

(b) *with coordinate axes tangent and normal to  $E$  at any point  $p$ , it admits a representation  $z = \phi(x, y)$ , where  $z$  is one-valued and has continuous partial derivatives, of order  $n$ , with respect to  $x$  and  $y$ , which are such that if  $q$  is any second point of  $E$ , and  $D^n \phi$  any definite one of these derivatives,*

$$|D^n \phi(q) - D^n \phi(p)| \leq D(t),$$

*where  $t$  is the projection of  $pq$  on the tangent plane at  $p$ , and where  $D(t)$  is a never decreasing function, independent of  $p$  and of the direction of  $pq$ , such that*

$$\int_0^\eta \frac{D(t)}{t} dt \quad (0 < \eta)$$

*is convergent;*

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\* For full details on these definitions, see Kellogg, *Foundations of Potential Theory*, Berlin, 1929, particularly p. 105.

(c) to every regular surface element  $E'$  contained in the interior of  $E$ , there corresponds a positive number  $a_1$ , such that the sphere of radius  $a_1$ , about any point of  $E'$ , contains no points of  $S$  other than those of  $E$ .

CONDITION  $B_n$ . The function  $U = U(x, y, z)$  is one-valued and continuous in  $R$ , and harmonic in the interior of  $R$ . Its values  $U(x, y, \phi(x, y))$  on  $E$ , the axes being tangent and normal to  $E$  at any point  $p$ , are subject to the condition imposed on  $\phi(x, y)$  in condition  $A_n(b)$ .

**Remarks.** The requirement of a representation  $z = \phi(x, y)$ , where  $z$  is single-valued and has continuous derivatives, for a single tangent-normal position of the axes, does not, of itself, assure such a representation for all such positions of the axes, even though  $E$  be arbitrarily flat.\* It does so, however, if, in addition to the requirement that every pair of normals make an acute angle with each other, we demand that the projection of  $E$  on one of its tangent planes be convex. It is convenience of application which has dictated the expression of the condition  $A_n(b)$  in this form. The condition  $A_n(c)$  excludes multiple boundary points on  $E$ . Otherwise  $S$  is unrestricted except that it must be bounded.  $A_n(a)$  is a consequence of  $A_n(b)$  for  $n \geq 2$ .

We may now formulate the main theorem of the paper.

**THEOREM III.** Let  $R$  satisfy Condition  $A_n$  and  $U$  Condition  $B_n$ . Let  $P$  be a point of  $R$  on the normal to  $E'$  at any of its points  $p$ . Then any given derivative of  $U$ , of order  $n$ , with respect to  $x, y$  and  $z$ , approaches a limit as  $P$  approaches  $p$  along the normal. If it is defined at  $p$  as equal to this limit, it is then a continuous function on  $E'$ , for unrestricted approach.

As the method of proof of this theorem is different, for  $n = 1$ , from that for  $n > 1$ , we consider the cases separately.

4. **Existence and continuity of the derivatives of the first order.** In order to infer properties of the boundary values of  $U$  on a sphere, internally tangent to  $E$ , from known properties of the boundary values on  $E$ , we shall need preliminary information on the rapidity of approach of  $U$  to its boundary values on  $E$ . This we shall obtain by means of a harmonic dominant function. It is the need of this function which largely accounts for the difference in the treatments of the derivatives of the first, and of higher orders.

First, however, we shall have need of a surface element  $E''$ , intermediate between  $E$  and  $E'$ . Let  $E''$  denote the portion of  $E$  whose points are distant not more than  $a_1/2$  from  $E'$ . Let  $a_2$  be the lower bound of the radii of curvature of  $E$ , and let  $a$  be the less of the two positive numbers  $a_1/8$  and  $a_2/2$ . Then  $a$  will have the properties

\* See *Foundations of Potential Theory*, loc. cit., p. 107, also Theorem VII, p. 108.

(a) any sphere of radius  $4a$  about a point of  $E''$  contains no points of  $S$  except those of  $E$ ,

(b) any sphere of radius  $4a$  about a point of  $E'$  contains no points of  $S$  except those of  $E''$ ,

(c) the sphere  $\sigma_i$ , of radius  $a$ , internally tangent to  $E$  at any point  $p$  of  $E'$ , will lie in the interior of  $R$  except at  $p$ , and the sphere  $\sigma_e$ , of radius  $a$ , externally tangent to  $E$  at any point  $p$  of  $E'$ , will be exterior to  $R$  except at  $p$ .

Let  $p$  be a point of  $E''$ . With axes in the tangent-normal position at  $p$ , we form the linear function

$$G_p = Ax + By,$$

$A$  and  $B$  being the derivatives at  $p$  of the boundary values of  $U$ , with respect to  $x$  and  $y$ , respectively. Then  $U_p = U - G_p$  is harmonic in  $R$ , and has boundary values on  $E$  which vanish, together with their derivatives of first order, at  $p$ . Moreover,  $G_p$ , and any of its derivatives, are uniformly bounded in  $R$ . As a consequence  $U_p$  is bounded in absolute value in  $R$ , by a constant  $M$ , independent of  $p$ . The law of the mean, and condition  $B_1$ , now yields, for any point  $q$  of  $E$ , not distant more than  $4a$  from  $E''$ ,

$$(3) \quad |U_p(q)| = |U_p'(\bar{q})|t = |U_p'(\bar{q}) - U_p'(p)|t \leq tD(\bar{t}) \leq tD(t),$$

the bars indicating appropriate mean points or values.

We now take up the harmonic dominant function. It is

$$W = \rho^\lambda P_\lambda(\cos \theta) \quad (0 < \lambda < 1),$$

where  $P_\lambda(u)$ ,  $u = \cos \theta$ , is that solution of Legendre's differential equation,

$$\frac{d}{du}(1-u^2)\frac{dP_\lambda}{du} + \lambda(\lambda+1)P_\lambda = 0,$$

which is regular at  $u=1$ , there assuming the value 1. The greatest root of this function in the interval  $(-1, +1)$ , if it has any, is negative. Under any circumstances, there is a positive number  $\alpha$ , which we may take less than  $\pi/2$ , such that for  $\cos(\pi/2 + \alpha) \leq u \leq 1$ ,  $P_\lambda(u)$  is positive, and in this interval,  $P_\lambda(u)$  is increasing. The last statement may be verified by forming the power series for  $P_\lambda(u)$  in  $z=1-u$ , which converges for  $|z| < 2$ , and has all its coefficients, after the constant term, negative.

We have, then, the following properties for  $W$ . It is continuous in the region

$$(4) \quad 0 \leq \rho, \quad 0 \leq \theta \leq \pi/2 + \alpha,$$

and harmonic in the interior. Its value at any point  $(\rho, \phi, \theta)$  lies between  $\rho^\lambda$  and its boundary values

$$W = C\rho^\lambda, \quad C = P_\lambda\left(\cos\left(\frac{\pi}{2} + \alpha\right)\right) = P_\lambda(-\sin \alpha) > 0.$$

Let us take, as origin of the spherical coördinates in terms of which  $W$  is expressed, a point  $p$  of  $E''$ , the axis from which  $\theta$  is measured being the inward normal. By the property (c) of the number  $a$ , all points of  $R$  within a distance  $2a \sin \alpha$  of  $p$  will lie in the region (4). The derivatives of first order of the boundary values of  $U$ , being continuous in a closed region, are uniformly bounded. The same is true for  $G_p$ . Hence by the first equation (3), the boundary values of  $U_p$  do not exceed, in absolute value, a uniform constant times  $W$ , in the sphere  $\sigma$  of radius  $2a \sin \alpha$  about  $p$ . On the portion of  $\sigma$  in  $R$ ,  $|U_p| \leq M$ , and as  $W$  has here a positive lower bound, there is a uniform constant,  $A$ , such that on the whole boundary of the portion of  $R$  in  $\sigma$ ,  $|U_p| \leq AW$ . As  $U_p$  and  $W$  are harmonic in this region, the inequality also holds in its interior. This leads to the inequality

$$(5) \quad |U_p(Q)| \leq A\overline{pQ}^\lambda,$$

valid, first, for any point  $Q$  of  $R$  in  $\sigma$ . But since  $|U_p|$  is bounded throughout  $R$ , and  $\rho^\lambda$  is an increasing function, the number  $A$  can be so chosen that the inequality holds throughout  $R$ .

Finally, since the derivatives of  $G_p$  are uniformly bounded in  $R$ , we have, for suitable  $B'$ ,

$$(6) \quad |G_p(Q) - G_p(p)| \leq B'\overline{pQ},$$

and hence, combining (5) and (6),

$$(7) \quad |U(Q) - U(p)| \leq B''\overline{pQ}^\lambda,$$

first, for  $\overline{pQ} \leq 1$ , and then for any  $Q$  in  $R$ ,  $B''$  being a constant independent of  $p$ . But (6) holds, if, without changing the linear function  $G_p$ , we substitute for the argument point  $p$ , any other point  $q$  of  $E''$ , and the same substitution may be made in (7). Hence we have, on combining the inequalities (6) and (7), thus altered,

$$(8) \quad |U_p(Q) - U_p(q)| \leq B\overline{qQ}^\lambda,$$

where  $q$  is any point of  $E''$ ,  $Q$  any point of  $R$ , and  $B$  is a constant, independent of  $p$ .

This result is valid for any  $\lambda$  in the open interval  $(0, 1)$ , the constant  $B$  depending, in general, on  $\lambda$ . For immediate purposes, we shall assign to  $\lambda$  a value greater than  $1/2$ .

5. Completion of the proof of the theorem for  $n=1$ . Let  $p$  now denote any point of  $E'$ ,  $\sigma$ , the sphere of radius  $a$ , internally tangent to  $E'$  at  $p$ , and  $Q$



a point of the lower half of the surface of  $\sigma$ . We wish to know that  $Q$  is on a normal to  $E$  at a point of  $E''$ ; for although the inequality (8) could be used without this knowledge, it will be useful later. By the properties (b) and (c) of the number  $a$ , we know that  $E$  lies between  $\sigma_i$  and the sphere  $\sigma_s$  externally tangent to  $E'$  at  $p$ , until it passes out of the sphere of radius  $4a$  about  $p$ . Hence  $E$  must cut the sphere  $\Sigma$ , through  $Q$ , and tangent to  $\sigma_s$  at the extremity of the radius which points toward  $Q$ . There are therefore points of  $E$  within a distance  $d$  of  $Q$ , where  $d$  is the diameter of  $\Sigma$ . We may find an appraisal for  $d$  by the cosine law of trigonometry. If  $\theta$  is the angle between the radii of  $\sigma_i$  to  $p$  and  $Q$ ,

$$(a + d)^2 = a^2 + (2a)^2 - 2(2a)a \cos \theta, \text{ or } d(2a + d) = 4a^2(1 - \cos \theta),$$

so that

$$d = \frac{8a^2 \sin^2 \frac{\theta}{2}}{2a + d} \leq 4a \sin^2 \frac{\theta}{2}.$$

Since  $Q$  is on the lower half of  $\sigma_i$ ,  $\theta \leq \pi/2$ , and  $d \leq 2a$ . If  $q$  is the point\* of  $E$  nearest  $Q$ , its distance from  $Q$  cannot exceed  $d$ , since, as we have seen, there are points of  $E$  within  $\Sigma$ . Hence  $\overline{pq} \leq \overline{pQ} + d \leq 2^{1/2}a + 2a < 4a$ , and by the property (b) of  $a$ ,  $q$  therefore lies on  $E''$ .

We now use the inequality (8). Since  $\overline{qQ} \leq d \leq a\theta^2$ , this yields

$$(9) \quad |U_p(Q) - U_p(q)| \leq Ba^\lambda \theta^{2\lambda}.$$

On the other hand, since  $t = a \sin \theta \leq a\theta$ , (3) yields

$$(10) \quad |U_p(q)| \leq a\theta D(a\theta).$$

Combining the inequalities (9) and (10), we see that the values  $U_p(Q)$  on the surface of  $\sigma_i$  are subject to the inequality

$$|U_p(a, \phi, \theta)| \leq \theta(Ba^\lambda \theta^{2\lambda-1} + aD(a\theta)) \quad (\theta \leq \pi/2).$$

It follows that the hypotheses of Theorems I and II are in force, since  $2\lambda - 1 > 0$ . Accordingly, the derivative of  $U_p$ , in any fixed direction, approaches a limit at  $p$  along the normal, and this, uniformly as to  $p$ . As the derivatives of  $G_p$  are bounded, uniformly as to  $p$ , the derivative of  $U$  itself approaches limits on  $E'$  along normals, uniformly. As the derivative is continuous in the interior of  $R$ , we infer that the same limits are approached for unrestricted approach of the argument point to the boundary. The assignment of these limiting values to the derivative, as values on  $E'$ , therefore

\* Or any of them, in case there are more than one. A similar comment applies at several points in the sequel.

makes the derivative continuous at the points of  $E'$ . Theorem III is thus proved for  $n = 1$ .

**Remarks on Condition  $B_1$ .** It is known\* that continuity of the boundary values, say on a circle, of the real part of a function of a complex variable, analytic in the circle, is not sufficient for the continuity of the boundary values of the conjugate function. We may conclude, by an integration, and by noting that a harmonic function of  $x$  and  $y$  may also be considered a harmonic function of  $x$ ,  $y$  and  $z$ , that something stronger than mere continuity must be required of the derivatives of the boundary values of  $U$  if we are to have continuous normal derivatives. The condition selected, although somewhat conditioned by the proof, is a fairly liberal one. It is clearly less restrictive than a Hölder condition on the derivatives:

$$|U'(q) - U'(p)| \leq At^\lambda \quad (0 < \lambda < 1).$$

In fact, if merely

$$|U'(q) - U'(p)| \leq A/[\log^\alpha(k/t)] \quad (\alpha > 1),$$

where  $k$  exceeds the maximum value  $t$  assumes, the function on the right will be seen to have the properties required of  $D(t)$  in Condition  $B_1$ .

**6. The derivatives of harmonic functions at interior points.** **Analytic character.** We shall need bounds for the derivatives of  $U$  at interior points of  $R$ . We may obtain these by applying a familiar inequality. Let  $V$  be harmonic in the sphere of radius  $c$  about  $P$ , and have there the upper and lower bounds  $M$  and  $m$ . Then if  $DV$  denote the derivative of  $V$  in any given direction, its value at  $P$  is subject to the inequality†

$$(11) \quad |DV| \leq \frac{3}{4c}(M - m),$$

or, in terms of the upper bound  $M$  of the absolute value of  $V$  on the sphere,

$$(12) \quad |DV| \leq \frac{3M}{2c}.$$

If  $V$  is defined in a region  $R$ , and  $M$  is the maximum of  $|V|$  in  $R$ ,  $c$  may be understood as the distance from  $P$  to the nearest boundary point of  $R$ .

We next seek a bound for the absolute value of the derivative  $D^2V$  of  $DV$  in any given direction, by applying (12) to a sphere of radius  $uc$  about  $P$ ,  $0 < u < 1$ . We have

\* See, for instance, Kellogg, *Potential functions on the boundary of their regions of definition*, these Transactions, vol. 9 (1908), p. 39, footnote ‡.

† See, for instance, *Foundations of Potential Theory*, loc. cit., p. 227.

$$|D^1V| \leq \frac{3}{2uc} \frac{3M}{2(1-u)c},$$

$(1-u)c$  being the distance from the sphere to the nearest point of the boundary of  $R$ . The result holds for any  $u$  in the given interval, and is closest when  $u = 1/2$ . It then gives

$$|D^2V| \leq \left(\frac{3}{2}\right)^2 M \frac{2^2}{c^2},$$

$c$  being again the distance from  $P$  to the nearest boundary point. Continuing in this way, we find for any derivative of  $V$ , of order  $n$ ,

$$(13) \quad |D^nV| \leq \left(\frac{3}{2}\right)^n M \frac{n^n}{c^n},$$

as we proceed to verify, by induction.

Assuming the formula (13), let us find, by means of (12), a bound for the value at  $P$  of the derivative in any given direction, of the harmonic function  $D^nV$ . On the sphere of radius  $uc$  about  $P$ , the absolute value of  $D^nV$  does not exceed

$$\left(\frac{3}{2}\right)^n M \frac{n^n}{(1-u)^n c^n}.$$

Using this bound in (12), and replacing  $c$  by  $uc$  in that inequality, we find

$$|D^{n+1}V| \leq \left(\frac{3}{2}\right)^{n+1} M \frac{n^n}{c^{n+1}} \frac{1}{u(1-u)^n}.$$

We choose  $u$  so that the last factor will take its least value,

$$u = \frac{1}{n+1}, \quad \frac{1}{u(1-u)^n} = \frac{(n+1)^{n+1}}{n^n}.$$

The inequality for  $D^nV$  thus obtained coincides with that given by the formula (13) when  $n$  is there replaced by  $n+1$ . As it is valid for  $n=1$ , (13) therefore holds generally.

As  $n^n \leq n!e^n$ , this inequality may be given the form

$$(14) \quad |D^nV| \leq \left(\frac{3e}{2c}\right)^n M n!.$$

Suppose  $V$  be developed in a Taylor series about the interior point  $P$  of  $R$ , with remainder. It will be found, by means of this bound for the derivatives of  $V$ , that for points whose distance from  $P$  is less than  $c/4$ , the remain-

der after the terms of degree  $n$  approaches 0 as  $n$  becomes infinite. The infinite series therefore converges to  $V$  in this neighborhood of  $P$ . We thus have a simple proof that  $V$  is analytic at any interior point of  $R$ .

However, the purpose for which the inequality (14) was derived was the study of the derivatives of higher order of Poisson's integral. The factor of the integrand which concerns us is

$$g = g(x, y, z) = \frac{a^2 - \rho^2}{r^3},$$

where  $r$  is measured from the point  $Q(\xi, \eta, \zeta)$  of the surface of the sphere of radius  $a$  about the origin of coördinates  $O$  to the point  $P(x, y, z)$  in the sphere, and where  $\rho$  is the distance  $OP$ . As it stands, bounds for  $g$ , which is harmonic throughout space, except at  $Q$ , are not evident, at least not in a form adapted to our needs. However, if we write  $\psi$  for the angle  $PQO$ , we have  $\rho^2 = a^2 + r^2 - 2ar \cos \psi$ , so that  $g$  becomes

$$g = \frac{2a \cos \psi}{r^2} - \frac{1}{r},$$

the terms on the right being harmonic except at  $Q$ , and being bounded in absolute value, at a distance  $r$  from  $Q$ , by  $2a/r^2$ , and by  $1/r$ , respectively.

If, now, we replace  $D^2V$  by  $(2a \cos \psi)/r^2$ , and, correspondingly,  $M$  by  $2a/3^2$ , (14), with  $n$  replaced by  $n+2$ , becomes

$$\left| D^n \frac{2a \cos \psi}{r^2} \right| \leq \left( \frac{3e}{2r} \right)^{n+2} \frac{2a}{3^2} (n+2)!.$$

Similarly, if we replace  $DV$  by  $1/r$ , and  $M$  by  $2/3$ , (14), with  $n$  replaced by  $n+1$ , becomes

$$\left| D^n \frac{1}{r} \right| \leq \left( \frac{3e}{2r} \right)^{n+1} \frac{2}{3} (n+1)!.$$

Combining these results, we have, for all points in the sphere, since there  $r \leq 2a$ ,

$$(15) \quad |D^n g| \leq 10a \left( \frac{3e}{2} \right)^n \frac{(n+2)!}{r^{n+2}}.$$

**7. The derivatives of order  $n$  of Poisson's integral.** We write Poisson's integral in the form

$$U = \frac{a}{4\pi} \int_0^\pi \int_0^{2\pi} U(a, \phi, \theta) g \sin \theta d\phi d\theta,$$

so that for  $\rho < a$ , any of the partial derivatives of order  $n$  with respect to  $x$ ,  $y$  and  $z$ , may be written

$$(16) \quad D^n U = \frac{a}{4\pi} \int_0^\pi \int_0^{2\pi} U(a, \phi, \theta) D^n g \sin \theta d\phi d\theta.$$

We regard this derivative as reckoned at a point of the polar axis  $\theta = 0$ , and divide the integral with respect to  $\theta$  into two parts, one from 0 to  $\eta$ , and one from  $\eta$  to  $\pi$ , where  $0 < \eta \leq \pi/2$ . Thus

$$D^n U = J_1 + J_2,$$

where  $J_2$ , for fixed  $\eta$ , is analytic in  $\rho$  at  $\rho = a$ , and where

$$(17) \quad \begin{aligned} |J_1| &\leq \frac{a}{4\pi} \int_0^\eta \int_0^{2\pi} |U(a, \phi, \theta)| |D^n g| \sin \theta d\phi d\theta \\ &\leq C_n \int_0^\eta \frac{f(\theta) d\theta}{\theta^{n+1}}. \end{aligned}$$

We have here used the fact, that for  $\theta \leq \pi/2$ ,  $\theta/r \leq \pi/(2a)$ , and have employed the abbreviations

$$C_n = \frac{5\pi^2}{4} \left( \frac{3\pi e}{4a} \right)^n (n+2)!, \quad f(\theta) = \frac{1}{2\pi} \int_0^{2\pi} |U(a, \phi, \theta)| d\phi.$$

The reasoning used to establish Theorem I now yields

**THEOREM IV.** *Let  $U$  be harmonic in the sphere of radius  $a$ , and be given by Poisson's integral with the bounded integrable boundary values  $U(a, \phi, \theta)$ . Let the average of the absolute value of this boundary function on parallel circles,*

$$f(\theta) = \frac{1}{2\pi} \int_0^{2\pi} |U(a, \phi, \theta)| d\phi,$$

*be subject to the requirement that*

$$(18) \quad \int_0^\eta \frac{f(\theta) d\theta}{\theta^{n+1}} \quad (0 < \eta)$$

*be convergent. Then any of the partial derivatives of order  $n$  of  $U$ , with respect to  $x$ ,  $y$  and  $z$ , at the point  $P$  of the polar axis, approaches a limit as  $P$  approaches the surface of the sphere along this axis. Moreover, the approach is uniform for any class of boundary functions which are uniformly bounded in absolute value, and for which the integral (18) approaches 0 uniformly with  $\eta$ .*

As remarked in connection with Theorem I, this result is broader than is at first apparent. For, provided that merely the derivatives of the boundary

values, of order  $n$ , have differences at neighboring points which approach 0 sufficiently rapidly with the distance between the points, the condition (18) may be brought to fulfillment by the subtraction from  $U$  of a suitable harmonic polynomial. We shall revert to this point in the next section.

8. A lemma on osculating harmonic polynomials. We now consider the existence of the polynomials, mentioned at the close of the last section, which broaden the scope of Theorem IV.

LEMMA. *Let the region  $R$  be subject to condition  $A_n$ , and the function  $U$  to condition  $B_n$ . We assume, moreover, that the derivatives of  $U$  of order  $n-1$  exist as limits on  $E$ , and are continuous there. Then, corresponding to each point  $p$  of  $E$ , there exists a harmonic polynomial,  $G_p$ , of degree  $n$ , such that*

$$U_p = U - G_p$$

*vanishes at  $p$ , together with all its derivatives of orders  $1, 2, \dots, n-1$ , and further, such that the derivatives of order  $n$  of its boundary values on  $E$  vanish at  $p$ . The values of  $G_p$ , and of its derivatives, are bounded in  $R$ , uniformly as to  $p$ .*

Taking the axes in the tangent-normal position at  $p$ , let  $G_{n-1,h}$  denote the sum of the terms of degree less than  $n$  in the development of  $U$  in spherical harmonics about the point  $P(0, 0, h)$  in the interior of  $R$ . As  $h \rightarrow 0$ , this harmonic polynomial approaches a limit  $G_{n-1}$ , since its coefficients, which are binomial coefficients times the derivatives of  $U$  of order  $n-1$  and lower, are continuous at the points of  $E$ . As these coefficients are subject to the equations which make  $G_{n-1,h}$  harmonic, these equations are satisfied in the limit, and so  $G_{n-1}$  is also harmonic. Thus  $U - G_{n-1}$  is harmonic in  $R$ , and vanishes, together with its derivatives of order  $n-1$  and lower, at  $p$ .

The derivatives of the boundary values of  $U - G_{n-1}$  of the same orders also vanish at  $p$ , while those of order  $n$  are the same as those of  $U$ . We form a homogeneous harmonic polynomial of order  $n$  as follows. We start with

$$f(x, y) = \frac{1}{n!} \left[ \left\{ x \frac{\partial}{\partial \xi} + y \frac{\partial}{\partial \eta} \right\}^n U(\xi, \eta, \phi(\xi, \eta)) \right]_{\xi=\eta=0},$$

whose derivatives of order  $n$  coincide at  $p$  with those of  $U$ ; from it, we form the homogeneous harmonic polynomial

$$H_n = f(x, y) - \frac{\nabla^2 f}{2!} z^2 + \frac{\nabla^2 \nabla^2 f}{4!} z^4 - \dots,$$

$\nabla^2 f$  denoting, as usual, the Laplacian of  $f$ . Because of the special position of the axes,  $z$  and its partial derivatives with respect to  $x$  and  $y$  vanish at  $p$ , so that the derivatives of the boundary values of  $H_n$  of order  $n$  reduce to those

of the first term at  $p$ , and thus to those of  $U$ . Because of the continuity of the derivatives of  $U$  and of  $\phi$  on the closed set  $E$ , the coefficients of  $G_{n-1}$  and  $H_n$  are bounded, uniformly as to  $p$ , and hence so are its values and those of its derivatives in  $R$ . Thus

$$G_p = G_{n-1} + H_n$$

has the properties required in the lemma.

It may be noted that  $G_p$ , although uniquely determined by the procedure for setting it up, is not uniquely determined by the properties enunciated in the lemma. Thus, if  $\psi(x, y)$  is any homogeneous polynomial of degree  $n-1$ , the harmonic polynomial

$$\psi(x, y)z - \frac{\nabla^2 \psi}{3!} z^3 + \frac{\nabla^2 \nabla^2 \psi}{5!} z^5 - \dots$$

may be added to  $G_p$  without impairing the requisite properties.

9. **Proof of the theorem for the derivatives of order  $n > 1$ .** The proof of Theorem III, for  $n > 1$ , is essentially a proof by induction, although, as we shall see, the case  $n=2$  occupies a somewhat special position. We therefore begin by noting that the conditions  $A_n$  and  $B_n$  imply the conditions  $A_{n-1}$  and  $B_{n-1}$ . Thus, since the derivatives of  $\phi$  of order  $n$  are continuous functions of the coördinates  $x$  and  $y$  and of the position of  $p$  in a closed region of these variables, they are uniformly bounded in absolute value. This means that the difference quotients of the derivatives of order  $n-1$  are bounded, and accordingly the function  $D(t) = \text{const.} \times t$  will serve as the required dominant function for them. The situation is the same with the boundary values of  $U$ .

Let  $n$  denote an integer,  $n \geq 2$ . We assume that Theorem III has been proved for all smaller values of  $n$ . That is, we assume that the conditions  $A_n$  and  $B_n$  are in force, and that all the partial derivatives of  $U$  of orders 1, 2,  $\dots$ ,  $n-1$  exist as limits on a regular surface element  $E$ , and are continuous there. We shall identify this regular surface element with the  $E$  of the theorem so as not to multiply notations. A later remark will make clear that this is legitimate.

We consider the function  $U_p = U - G_p$  of the lemma, and take the axes in the usual tangent-normal position at  $p$ , a point of  $E'$ . We construct the sphere  $\sigma_i$ , internally tangent to  $E'$  at  $p$  of radius  $a$ . Let  $Q$  be a point of the lower half of the surface of  $\sigma_i$  and  $q$  the foot of a normal to  $E$  through  $Q$  (see §5). Then

$$(19) \quad \begin{aligned} U_p(Q) = U_p(q) + \frac{\partial U_p}{\partial \nu} \bigg|_q \overline{qQ} + \frac{1}{2} \frac{\partial^2 U_p}{\partial \nu^2} \bigg|_q \overline{qQ}^2 + \dots \\ + \frac{1}{(n-1)!} \frac{\partial^{n-1} U_p}{\partial \nu^{n-1}} \bigg|_{q_1} \overline{qQ}^{n-1}, \end{aligned}$$

where  $q_1$  is an interior point of the segment  $\overline{qQ}$ . This equation is legitimate, since the derivatives of  $U_p$  of order  $n-1$  exist.

To avoid a whole series of different notations, let us agree to denote by  $K$  any function which has a bound for its absolute value which may depend on  $R$ ,  $E'$ ,  $U$ , and  $n$ , but not on  $p$ , and similarly, let us denote by  $D(t)$  any function dominated by a function satisfying the requirements of condition  $A_n$  on  $D(t)$ . These notations may therefore mean different functions from time to time, or even in the same equation, but no difficulty will arise if this fact be kept in mind. Our object is to establish an equation

$$(20) \quad |U_p(Q)| = t^n D(t).$$

To do this, we develop the coefficients in (19), by Taylor's series with remainders, about the point  $p$ . For the first, since the derivatives of lower order vanish at  $p$ , we have

$$U_p(q) = \frac{1}{n!} \left. \frac{\partial^n U_p}{\partial t^n} \right|_{p_1} t^n,$$

where  $p_1$  is an appropriate mean point on  $E$ . But, by Condition  $B_n$ ,

$$\left| \frac{\partial^n U_p}{\partial t^n} \right|_{p_1} = \left| \frac{\partial^n U_p}{\partial t^n} \right|_{p_1} - \left. \frac{\partial^n U_p}{\partial t^n} \right|_p \leq D(t).$$

Hence

$$(21) \quad |U_p(q)| = t^n D(t).$$

Passing to the second term in the development (19), we have

$$(22) \quad \begin{aligned} \left. \frac{\partial U_p}{\partial \nu} \right|_q &= \left. \frac{\partial U_p}{\partial \nu} \right|_p + \left. \frac{\partial^2 U_p}{\partial t \partial \nu} \right|_p t + \cdots + \frac{1}{(n-2)!} \left. \frac{\partial^{n-1} U_p}{\partial^{n-2} t \partial \nu} \right|_{p_1} t^{n-2}, \\ &= K t^{n-2}, \end{aligned}$$

since the derivatives of  $U_p$  of order  $n-1$  are uniformly bounded on  $E$  and vanish at  $p$ . Since  $\overline{qQ} = Kt^2$ , as we say at the beginning of §5, we have for this second term in (19),

$$\left. \frac{\partial U_p}{\partial \nu} \right|_{\overline{qQ}} = K t^n.$$

Similar considerations show that the later terms in (19) are bounded functions times  $t^{n+1}$ , the order with respect to  $t$  increasing, at each step, by unity. We thus have the preliminary result

$$(23) \quad U_p(Q) = K t^n.$$



While this is not as sharp an appraisal as the needed one (20), it will serve us in attaining that goal. The difficulty is obviously with the second term in (19). If  $n=2$ , the argument in the derivative must be a mean point  $q_1$ , since the second is then the final term. It is in this sense that the case  $n=2$  is special. We see, then, that if we show that

$$(24) \quad \left| \frac{\partial U_p}{\partial \nu} \right|_q = t^{n-2} D(t) \text{ for } n > 2, \text{ and } \left| \frac{\partial U_p}{\partial \nu} \right|_{q_1} = D(t) \text{ for } n = 2,$$

the desired equation (20) will be assured.

*The case  $n > 2$ .* We shall specialize the axes still further by taking the  $x$ -axis along the projection of  $pq$ , so that  $x=t$ . We have, then, as we see by (22), to prove that

$$(25) \quad \left| \frac{\partial U_p}{\partial \nu} \right|_q / t^{n-2} = \frac{1}{(n-2)!} \left| \frac{\partial^{n-1} U_p}{\partial x^{n-2} \partial \nu} \right|_{p_1} = D(t).$$

We may, however, replace the mean argument point  $p_1$  by  $q$ , since the projection on the tangent plane of  $pp_1$  is not greater than that of  $pq$ , or  $t$ . In (25)

$$\frac{\partial U_p}{\partial \nu} = \left[ -\frac{\phi_x}{w} \frac{\partial U_p}{\partial x} - \frac{\phi_y}{w} \frac{\partial U_p}{\partial y} + \frac{1}{w} \frac{\partial U_p}{\partial z} \right]_{z=\phi(x,y)}, \quad w = (1 + \phi_x^2 + \phi_y^2)^{1/2}.$$

By Leibnitz' rule for products,

$$\begin{aligned} \frac{\partial^{n-2}}{\partial x^{n-2}} \left[ -\frac{\phi_x}{w} \frac{\partial U_p}{\partial x} \right]_{z=\phi(x,y)} \\ = \sum_0^{n-2} \binom{n-2}{i} \frac{\partial^{n-2-i}}{\partial x^{n-2-i}} \left( -\frac{\phi_x}{w} \right) \frac{\partial^i}{\partial x^i} \left[ \frac{\partial U_p}{\partial x} \right]_{z=\phi(x,y)}. \end{aligned}$$

For all values of  $i$  less than  $n-2$ , the second factor in each term vanishes at  $p$ , and has bounded derivatives of the first order with respect to  $x$  and  $y$ , while the first factor is bounded. These terms are therefore of the form  $Kt$ . For  $i=n-2$ , the second factor is bounded, while the first one vanishes at  $p$ , and has bounded derivatives, and so is also of the form  $Kt$ , because of the special position of the axes. The same is true of

$$\frac{\partial^{n-2}}{\partial x^{n-2}} \left[ -\frac{\phi_y}{w} \frac{\partial U_p}{\partial y} \right]_{z=\phi(x,y)}.$$

Finally, we see in a similar way that

$$\begin{aligned} \frac{\partial^{n-2}}{\partial x^{n-2}} \left[ \frac{1}{w} \frac{\partial U_p}{\partial z} \right]_{z=\phi(x,y)} &= Kt + \frac{1}{w} \frac{\partial^{n-2}}{\partial x^{n-2}} \left[ \frac{\partial U_p}{\partial z} \right]_{z=\phi(x,y)} \\ &= Kt + \frac{1}{w} \left[ \frac{\partial^{n-1} U_p}{\partial x^{n-2} \partial z} \right]_{z=\phi(x,y)} \end{aligned}$$

since the difference of the derivatives on the right is a sum of terms each of which is a bounded function times a derivative of  $U_p$  of lower order, or times a power of  $\phi_x$ .

As  $Kt$  is a function  $D(t)$ , and as the sum of two such functions belongs to the same class, the establishment of the equation (25) is thus reduced to proving that

$$(26) \quad |V(q)| = |D^{n-1}U_p|_q = \left| \frac{\partial^{n-1}U_p}{\partial x^{n-2}\partial z} \right|_q = D(t).$$

Since  $V(p) = 0$ , the problem is to determine how rapidly the function  $V(q)$  approaches its value at  $p$  as  $q \rightarrow p$ . We may proceed as follows. Let  $P$  and  $Q$  be points of  $R$  on the normals to  $E$  at  $p$  and  $q$ , respectively, with  $\overline{pP} = \overline{qQ} = \delta > 0$ . We compare  $V(P)$  with  $V(p)$ ,  $V(Q)$  with  $V(q)$ , and then  $V(P)$  with  $V(Q)$ .

For the first, we apply Poisson's integral to  $U_p$ , using the sphere  $\sigma_i$  tangent to  $E$  at  $p$ . By (16), we have

$$V(P) = \frac{a}{4\pi} \int_0^\pi \int_0^{2\pi} U_p(a, \phi, \theta) D^{n-1}g \sin \theta d\phi d\theta,$$

and if  $p'$  is a point between  $p$  and  $P$ , distant  $\rho'$  from the center of  $\sigma_i$ ,

$$V(p') - V(P) = \frac{a}{4\pi} \int_\rho^{\rho'} \int_0^\pi \int_0^{2\pi} U_p(P') \frac{\partial}{\partial \rho} D^{n-1}g \sin \theta d\phi d\theta d\rho,$$

the integrand being continuous.  $P'$  is the point  $(a, \phi, \theta)$ . We break the integral with respect to  $\theta$  into two parts, the first over the interval  $(0, \pi/2)$ , and the second over the interval  $(\pi/2, \pi)$ . In the first,  $U_p(P') = Kt^n = K\theta^n$ , by (23). In the second,  $|U_p(P')| \leq M$ , and  $r > a$ , if  $pP < a$ , as we have already implicitly assumed. In both integrals,

$$\frac{\partial}{\partial \rho} D^{n-1}g = \frac{K}{r^{n+2}},$$

by (15). Accordingly, we may write

$$V(p') - V(P) = J_1 + J_2,$$

where

$$|J_2| \leq \frac{a}{4\pi} \int_\rho^{\rho'} \int_{\pi/2}^\pi \int_0^{2\pi} \frac{MK}{a^{n+2}} d\phi d\theta d\rho = K(\rho' - \rho) = K(a - \rho),$$

and

$$\begin{aligned}
 |J_1| &\leq \frac{a}{4\pi} \int_{\rho}^{\rho'} \int_0^{\pi/2} \int_0^{2\pi} K \theta^n \frac{K}{r^{n+2}} \theta d\phi d\theta d\rho \\
 &= K \int_{\rho}^{\rho'} \int_0^{\pi/2} \frac{1}{r} d\theta d\rho,
 \end{aligned}$$

where, in the last step, we have used the inequality  $\theta/r \leq \pi/(2a)$ .

For the inner integral, we find

$$\begin{aligned}
 \int_0^{\pi/2} \frac{1}{r} d\theta &\leq 2^{1/2} \int_0^{\pi/2} \frac{\cos \frac{\theta}{2} d\theta}{\left[ (a - \rho)^2 + \left( 2(a\rho)^{1/2} \sin \frac{\theta}{2} \right)^2 \right]^{1/2}} \\
 &= \left( \frac{2}{a\rho} \right)^{1/2} \log \frac{(2a\rho)^{1/2} + (a^2 + \rho^2)^{1/2}}{a - \rho} \leq \frac{2}{a} \log \frac{2 \cdot 2^{1/2} a}{a - \rho},
 \end{aligned}$$

if  $\rho \geq a/2$ . Hence

$$J_1 = K \int_{\rho}^{\rho'} \log \frac{3a}{a - \rho} d\rho = K \left[ (a - \rho') \log \frac{a - \rho'}{3ae} - (a - \rho) \log \frac{a - \rho}{3ae} \right].$$

As  $V(p') \rightarrow V(p)$  as  $\rho' \rightarrow a$ , we find, therefore,

$$V(p) - V(P) = K \left[ (a - \rho) + (a - \rho) \log \frac{3ae}{a - \rho} \right] = K(a - \rho) \log \frac{a}{a - \rho}.$$

This gives, in terms of  $\delta = a - \rho$ , for  $\delta < a/2$ ,

$$(27) \quad V(p) - V(P) = K\delta \log \frac{a}{\delta}.$$

When we consider  $V(q) - V(Q)$ , we must first make sure that  $q$  is in the region for which (23), with  $p$  replaced by  $q$ , and  $Q$  by a point on the lower half of the corresponding sphere  $\sigma_i$ , is valid. But this is true because  $q$  is on  $E''$ , and distant from its edge at least  $4a - (2^{1/2} + 2)a = (2 - 2^{1/2})a$ , by §5. We have also to consider the effect of adding to  $U_p$  the harmonic polynomial  $G_p - G_q$ . Since the derivatives of  $G_p$  are all bounded in  $R$ , uniformly as to  $p$ , this addition affects  $V(q) - V(Q)$  only by adding a term  $K\overline{qQ} = K\delta$ . Hence we infer also that

$$(28) \quad |V(q) - V(Q)| = K\delta \log \frac{a}{\delta}.$$

When it comes to comparing  $V(P)$  with  $V(Q)$ , we connect  $P$  and  $Q$  by a curve  $\gamma$ , never nearer than  $\delta$  to the boundary of  $R$ . We have, then,

$$V(Q) - V(P) = \int_{sP}^{sQ} \frac{\partial V}{\partial s} ds.$$

Let us take for  $\gamma$  the locus of the centers of the spheres of radius  $\delta$ , internally tangent to  $E$  at the points where  $E$  is cut by the  $(x, z)$ -plane, i.e., at the points of the curve  $x = x, y = 0, z = \phi(x, 0)$ . Then  $\gamma$  is given by

$$\xi = x - \frac{\phi_x}{w} \delta, \quad \eta = -\frac{\phi_y}{w} \delta, \quad \zeta = \phi + \frac{1}{w} \delta.$$

We find that the derivatives of  $\xi, \eta$ , and  $\zeta$  with respect to  $x$  are uniformly bounded, and hence the length of  $\gamma$  is a bounded function times the  $x$ -coördinate of  $q$ , or  $t$ . Moreover, the harmonic function  $V$  is uniformly bounded in the portion of  $R$  swept out by the spheres of radius  $4a$  about the points of  $E'$ . Let  $B$  be a bound for its absolute value in this region. Then at the points of  $\gamma$ , by (11),

$$\left| \frac{\partial V}{\partial s} \right| \leq \frac{3}{2\delta} B.$$

Accordingly, we have

$$(29) \quad |V(Q) - V(P)| \leq K \frac{t}{\delta}.$$

We now combine the results (27), (28), (29), writing  $\delta = t^{1/2}$ . Then, for  $t < a^2/4$ ,

$$V(q) - V(p) = K t^{1/2} \log \frac{a^2}{t} + K t^{1/2} = K t^\lambda,$$

if  $\lambda$  is any number between 0 and  $1/2$ . There is no difficulty in extending such a relation to values of  $t$  greater than  $a^2/4$ , since  $V(q)$  is bounded on  $E$ . As  $K t^\lambda$  has the properties required of  $D(t)$ , the equation (26), and with it the first equation (24), is established.

*The case  $n = 2$ .* We have to show that

$$|V(q_1)| = \left| \frac{\partial U_p}{\partial \nu} \right|_{q_1} = D(t).$$

If  $\delta_1$  is the distance from  $q$  to  $q_1$ , we find, as before,

$$|V(q_1) - V(q)| = K \delta_1 \log \frac{a}{\delta_1} = K \delta \log \frac{a}{\delta}.$$

But the preceding considerations have proved that  $V(q) - V(p) = D(t)$ . Accordingly

$$|V(q_i)| = D(t),$$

and the second equation (24) is established. But this, it will be recalled, is sufficient for the equation (20), which we set out to establish.

The proof of Theorem III is now readily completed. The equation (20) leads at once to

$$|U_p(Q)| = \theta^n D(\theta),$$

so that the hypothesis of Theorem IV is fulfilled by the boundary values of  $U_p$  on  $\sigma_i$ , and accordingly, the derivatives of order  $n$  of  $U_p$  approach limits at  $p$  along the normal. Moreover, the approach is uniform as to  $p$ , and these derivatives, rightly defined on  $E'$ , are then continuous in  $R$  at the points of  $E'$ .

It remains only to justify the assumption that the derivatives of order  $n-1$  and lower were continuous at the points of  $E$ . We may interpolate a set of surface elements between  $E$  and  $E'$ , each interior to the preceding. On the first, the derivatives of first order are continuous, on the second, those of second order are continuous, and so on. Letting the  $(n-1)$ th play the rôle of  $E$  in the above proof, we have established the existence and continuity of the derivatives of the  $n$ th order on  $E'$ . Theorem III is thus completely proved.

10. Hölder conditions on  $U$ . We shall consider, in this section, Hölder conditions on  $U$  itself, and in the next, Hölder conditions on the derivatives. We assume

CONDITION  $A_\lambda$ . This is obtained from Condition  $A_1$ , with (a) omitted, and with  $D(t)$  specialized so as to take the form  $A t^\lambda$ ,  $0 < \lambda < 1$ , so that

$$|D\phi(q) - D\phi(p)| \leq A t^\lambda.$$

CONDITION  $B_\lambda$ .  $U$  is continuous in  $R$ , and harmonic in the interior of  $R$ , and if  $p$  and  $q$  are any two points of  $E$ ,

$$|U(q) - U(p)| \leq A t^\lambda.$$

We then have the theorem

THEOREM V. If  $R$  is subject to Condition  $A_\lambda$ , and  $U$  to Condition  $B_\lambda$ , there is a region  $R'$ , containing all the points of  $R$  in a neighborhood of  $E'$ , and a constant  $B$ , such that for any two points  $P$  and  $Q$  of  $R'$ ,

$$|U(Q) - U(P)| \leq B r^\lambda, \quad r = \overline{PQ}.$$

We may choose  $R'$  at once as those points of  $R$  whose distances from  $E'$  do not exceed  $a$  (see §4). Reverting to the dominant harmonic function  $W = \rho^\lambda P_\lambda(\cos \theta)$  of §4, we take the origin of the system of spherical coör-

dinates at any point  $p$  of  $E''$ , with the axis of  $\theta$  in the direction of the inward normal. Then a portion of  $E$  in the neighborhood of  $p$  lies in the region (4). For, with axes of cartesian coördinates in the usual tangent-normal position at  $p$ , we have, by Condition  $A_\lambda$ ,

$$|z| = |\phi_z(q_1)x + \phi_y(q_1)y| \leq 2^{1/2}At^{1+\lambda},$$

while the boundary of (4) is given by

$$z = -\tan \alpha t.$$

Hence all points of  $R$  in a sphere about  $p$ , of radius not greater than  $[(\tan \alpha)/(2^{1/2}A)]^{1/\lambda}$ , lie in the region (4).

We conclude, as in §4, that there is a constant  $B'$ , independent of  $p$ , such that for any point  $p$  on  $E''$ , and any point  $Q$  in  $R$ ,

$$(30) \quad |U(Q) - U(p)| \leq B' \overline{pQ}^\lambda.$$

The problem is now to extend this inequality to points  $P$  in  $R'$ . If  $P$  is any point of  $R'$ , its distance from  $E'$  is not more than  $a$ , while the distance from  $E'$  of any point of  $S$  not in  $E''$  is at least  $4a$ . Hence any point  $P$  of  $R'$  is nearer to some point  $p$  of  $E''$  than to any other boundary point of  $R$ . Let  $p$  be the nearest point of  $E''$ , distant  $c$ , say, from  $P$ . Let  $\sigma$  denote the sphere of radius  $c/2$  about  $P$ . Then, by (30), the oscillation of  $U$  on  $\sigma$  does not exceed twice the maximum on  $\sigma$ , of  $B' \overline{pQ}^\lambda$ . Accordingly, by (11), the derivative  $DU$  of  $U$ , in any direction, at  $P$ , is subject to the inequality

$$(31) \quad |DU| \leq \frac{3B'(3c/2)^\lambda}{2(c/2)} = B''c^{\lambda-1}.$$

Now let  $P$  and  $Q$  be any two points of  $R'$ . We consider first the case in which  $r=PQ$  is less than the distance of the segment  $PQ$  from  $E''$ . Here, integrating along the segment  $PQ$ , we have

$$|U(Q) - U(P)| = \left| \int_0^r \frac{\partial U}{\partial s} ds \right| \leq B''c^{\lambda-1}r \leq B''r^\lambda.$$

On the other hand, if the length  $r$  of the segment  $PQ$  is greater than or equal to its distance from  $E''$ , let  $s$  be the point of  $E''$  nearest the segment. Then  $sQ$  and  $sP$  are not greater than  $2r$ , and (30) yields, when applied to the pairs of points  $s, Q$ , and  $s, P$ ,

$$|U(Q) - U(P)| \leq 2B'2^\lambda r^\lambda.$$

Hence if  $B$  denotes the larger of the two constants  $B''$  and  $2^{1+\lambda}B'$ , we have, for any two points  $P$  and  $Q$  in  $R'$ ,

$$|U(Q) - U(P)| \leq Br^\lambda.$$

Theorem V is thus proved.

**Remark.** The exponent  $\lambda$  has been confined to the open interval  $(0, 1)$ . For  $\lambda = 1$ , the Hölder condition becomes a Lipschitz condition, and such a condition on the boundary values of  $U$  does not imply a similar condition for neighboring interior points. This may be shown by an example. When the sphere to which Poisson's integral is applied becomes the infinite plane, we have the following representation of a function, harmonic to one side of this plane, and assuming the boundary values  $f$ :

$$U(P) = \frac{1}{2\pi} \iint f d\Omega,$$

where  $\Delta\Omega$  denotes the solid angle subtended at  $P$  by the element of surface  $\Delta S$  of the plane, the integral being extended over the infinite plane. Using cylindrical coördinates  $(\rho, \phi, z)$ , with origin in the plane, and  $z$ -axis normal to it, we consider the function defined by the boundary values  $f = \rho/(1 + \rho^2)$ , at points of the  $z$ -axis,  $z > 0$ . The evaluation of the integral gives, for such points,

$$U = \frac{z}{(1 - z^2)^{3/2}} \left[ \log \frac{1 + (1 - z^2)^{1/2}}{z} - (1 - z^2)^{1/2} \right],$$

and  $U$  therefore fails to have bounded difference quotients near the origin, although its boundary values do have.

11. **Hölder conditions on the derivatives of  $U$ .** The conditions which we here assume are  $A_{n+\lambda}$  and  $B_{n+\lambda}$ ; they are simply the conditions obtained from  $A_n$  and  $B_n$  by specializing the function  $D(t)$  to be of the form  $At^\lambda$  ( $0 < \lambda < 1$ ). As the definition of  $D(t)$  implies,  $A$  and  $\lambda$  are independent of  $p$  and of the direction of  $pq$ . We conclude by establishing

**THEOREM VI.** *If  $R$  is subject to Condition  $A_{n+\lambda}$  and  $U$  to Condition  $B_{n+\lambda}$ , then there is a region  $R'$ , containing all points of  $R$  in a neighborhood of  $E'$ , and a constant  $B$ , such that for any two points  $P$  and  $Q$  of  $R'$ ,*

$$|D^n U(Q) - D^n U(P)| \leq Br^\lambda, \quad r = \overline{PQ}.$$

Here as before,  $D^n U$  means any one of the derivatives of  $U$  of order  $n$  with respect to  $x, y$  and  $z$ , the axes of these coördinates being fixed.

By Theorem III, we know that the derivatives of order  $n$  of  $U$  exist and are continuous at the points of any closed surface element interior to  $E$ . We may infer that these derivatives are bounded in the region  $R''$  containing all points of  $R$  whose distances from  $E''$  do not exceed  $(2^{1/2} + 2)a$ , and no others.

Let  $p$  be any point of  $E''$ , and  $\sigma$ , the sphere of radius  $a$ , internally tangent

to  $E''$  at  $p$ . Let  $Q$  lie on the lower half of the surface of  $\sigma_i$ . Its distance from  $p$  is then not more than  $2^{1/2}a$ , and hence the nearest point of  $S$  to  $Q$  is in  $R''$ , and so on  $E$ . Call such a point  $q$ . Then (19) holds for the function  $U_p$ , defined in §8. We conclude, as in §9—except that the steps are much simplified by our present knowledge, by Theorem III, that the derivatives of  $U$  of order  $n$  are bounded in  $R''$ —that

$$(32) \quad U_p(Q) = Kt^{n+\lambda},$$

where we are again adopting the convention that  $K$  means any function whose absolute value has a bound independent of  $p$ , and of any other argument points.

From this, we infer, using the method of §9, and applying Poisson's integral in the sphere  $\sigma_i$ , that if  $D^n U$  is any given derivative of  $U$  of order  $n$ ,

$$(33) \quad D^n U_p(P) - D^n U_p(p) = K\delta^\lambda,$$

for any point  $P$  on the normal at  $p$ , distant  $\delta$  from  $p$ ,  $\delta \leq a$ . This latter restriction may, however, be dropped, as we have seen, provided we remain in  $R''$ . This leads, as in the preceding section, to

$$D^{n+1}U_p(P) = Kc^{\lambda-1},$$

where  $c$  is the distance from  $P$  to  $E$ , and  $P$  is on a normal to  $E$  at a point of  $E''$ , and in  $R''$ . Since all derivatives of  $G_p$  are bounded, uniformly as to  $p$ , in  $R''$ , the last equation yields

$$(34) \quad D^{n+1}U(P) = Kc^{\lambda-1}.$$

For  $R'$ , we take, as before, the set of all points of  $R$  whose distances from  $E'$  do not exceed  $a$ . Any point of  $R'$  is on a normal to  $E$  at some point of  $E''$ . Let  $P$  and  $Q$  be any two points of  $R'$ . As before, we have two cases to consider. If the distance  $r = PQ$  is not greater than the distance between the segment  $PQ$  and  $E''$ , we argue, as before, that

$$(35) \quad |D^n U(Q) - D^n U(P)| \leq Br^\lambda.$$

This is the desired result, established for this case.

If  $r$  is greater than the distance between the segment  $PQ$  and  $E''$ , new geometric considerations are needed. Because of the continuity of  $\phi_x$  and  $\phi_y$ , there corresponds to any positive angle  $\beta$ , a number  $b$ , such that if  $p$  and  $q$  are any two points of  $E$ , whose distance is not more than  $b$ , the normals to  $E$  at  $p$  and  $q$  make an angle not greater than  $\beta$ . We shall take  $\beta$  as the acute angle for which  $\sin(\beta/2) = 1/8$ . Let  $p$  and  $q$  be two points of  $E''$ , whose distance  $r$  does not exceed  $b$ ,  $b$  being further restricted, if necessary, so as not



to exceed  $a/4$ . Let  $P$  be in  $R''$ , on the normal to  $E''$  at  $p$ , and let  $Q$  be in  $R''$  and on the normal to  $E''$  at  $q$ , such that  $pP = qQ = 4r$ .

We find, then, that  $PQ \leq 2(4r) \sin (\beta/2) + r = 2r$ . Since  $r \leq b \leq a/4$ , the nearest point of  $E$  to  $P$  is  $p$ , and its distance is  $4r$ . Thus the whole segment  $\overline{PQ}$  is distant from  $E''$  at least as much as the length  $\overline{PQ}$ , and (35) is applicable. It gives

$$D^n U(Q) - D^n U(P) = K(2r)^\lambda = Kr^\lambda.$$

But, by (33), we have also

$$D^n U(P) - D^n U(p) = K(4r)^\lambda = Kr^\lambda,$$

and, similarly,

$$D^n U(Q) - D^n U(q) = Kr^\lambda.$$

Combining the last three equations, we have

$$(36) \quad D^n U(q) - D^n U(p) = Kr^\lambda.$$

The preliminary restriction that  $r \leq b$  may now be removed by the usual argument.

Now let  $P$  and  $Q$  be any two points of  $R'$  whose distance exceeds the distance of the segment  $\overline{PQ}$  from  $E''$ . There will be a point  $s$  of  $E''$  whose distance from  $\overline{PQ}$  is less than  $r$ , and therefore, whose distances from  $P$  and  $Q$  are less than  $2r$ . Hence the nearest points of  $E''$  to  $P$  and  $Q$ , which we call  $p$  and  $q$ , respectively, will be distant from  $P$  and  $Q$ , respectively, less than  $2r$ . The distance  $\overline{pq}$ , accordingly, cannot exceed  $5r$ . Applying the equation (33) to the pairs of points  $P, p$  and  $Q, q$  and the equation (36) to the points  $p, q$ , we obtain the inequality (35) for the second case. Here  $P$  or  $Q$  or both may lie on  $E''$ , and so be any points in the closed region  $R'$ . Theorem VI is thus established.

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